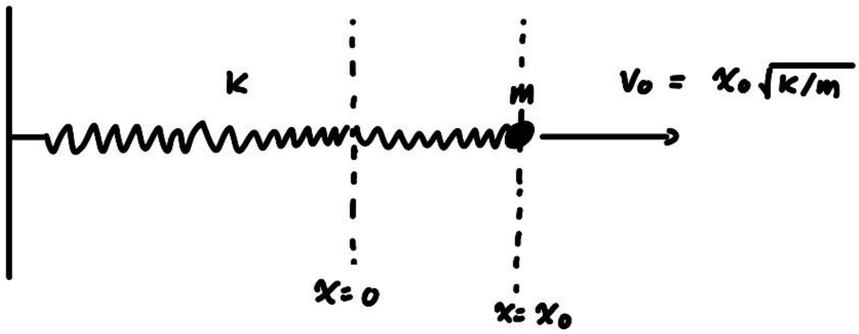
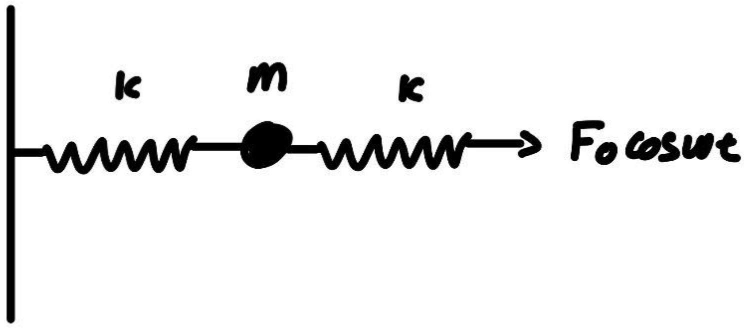
Sample Midterm Problems

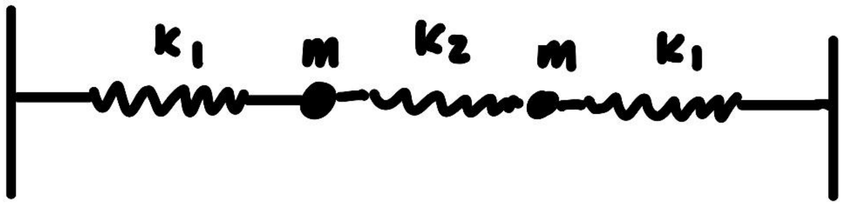
1. Suppose a mass m is placed in a potential modelled by the function $U(x) = ax^3 + bx^2 + U_0$ where $b > 0$.
   A. Verify that the particle is in equilibrium at $x = 0$
   B. Suppose the particle now undergoes small oscillations about $x = 0$ in the potential well. Derive a second-order Taylor Series Approximation of $U(x)$ at x close to 0.
   C. Based on the approximation in b), derive an ODE modelling the motion of the mass near $x = 0$ as simple harmonic motion. What is the angular frequency?



2. A mass m attached to a spring with constant k oscillates about $x = 0$ as shown in the diagram above. The mass is also subject to a damping force $F_d = -bv$.
   A. Write the ODE modelling the motion of the mass.
   B. Assume that the system is lightly damped. At time $t = 0$ the mass is at position $x = x_0$ and moving with velocity $\dot{x} = x_0\sqrt{\frac{k}{m}}$. Using the Ansatz for damped simple harmonic motion, solve for the phase difference δ and initial amplitude A_0. Make sure to define any new quantities you introduce in your solution.
   C. Derive the initial energy E_0 of the system in terms of A_0. (HINT: Directly use the results from part b)



3. A mass m is connected to two springs on either side with spring constants k. The mass is also subject to a damping force $F_d = -bv$.
   A. For what values of b is the system lightly damped?
   B. Suppose a driving force $F(t) = F_0 \cos \omega t$ is applied at the right-most spring. Write the ODE modelling the motion of the mass.
   C. Assuming the system is lightly damped and relaxes to a steady-state solution, what is the amplitude and phase as functions of driving frequency ω?
   D. Sketch the power resonance curve $\overline{P(\omega)}$, labelling frequencies where $\overline{P(\omega)} = \overline{P_{max}}$ and $\overline{P(\omega)} = \overline{P_{max}}/2$. Write the frequencies in terms of m, k, b. Also determine the value of $\overline{P_{max}}$.



4. Two masses (both m) are connected by three springs in the system shown above. The left and right springs have constant k_1 and the middle spring has constant k_2.
   A. Write the two coupled ODEs describing the motions of the two masses $x_1(t)$ and $x_2(t)$.
   B. Using the coupled ODEs in part a), write two independent uncoupled ODEs for q_1 and q_2. Then solve for $q_1(t)$ and $q_2(t)$ as functions of time. Set the phases $\phi_1 = \phi_2 = 0$.
   C. What are the angular frequencies corresponding to the normal modes of the system?
   D. Using $q_1(t)$ and $q_2(t)$, solve for $x_1(t)$ and $x_2(t)$.
   E. Describe the motion of the masses when they oscillate at the normal frequencies.